§16.3 Conservative Vector Fields –

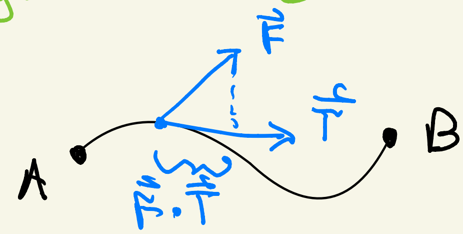
Recall: Line Integral (all equivalent)

$$\underbrace{\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds}_{\text{Tells Meaning}} = \underbrace{\int_a^b \vec{F}\cdot\vec{v}\,dt}_{\text{tells how to compute it}} = \int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_{\mathcal{C}} M\,dx + N\,dy + P\,dz$$

Tells Meaning
- Work Done
- Circulation
- "Amt of $\vec{F}$ pointing tangent to $\mathcal{C}$"

tells how to compute it

"View a parameterization $\vec{r}(t)$ as a coordinate system on $\mathcal{C}$"



We proved:

Thm: If $\vec{F} = \nabla f$, then $\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = f(B) - f(A)$

Said Differently: We can evaluate $\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds$ if we can find an "anti-derivative" f such that $\nabla f = \vec{F}$. (f is called a "potential" – If so, we say $\vec{F}$ Conservative)

I.e., then: $\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = \int_{\mathcal{C}} \nabla f\cdot\vec{T}\,ds = f(B) - f(A)$

Q1: Given $\vec{F}$, how do we know f exists st $\nabla f = \vec{F}$?

Q2: Given $\vec{F}$ has a potential, how do we find it?

Note: If $\vec{F}$ is conservative, so there is an f such that $\nabla f = \vec{F}$, then

indicates $\mathcal{C}$ is closed

$$\oint_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = 0 \qquad (f(B)-f(A)=0)$$

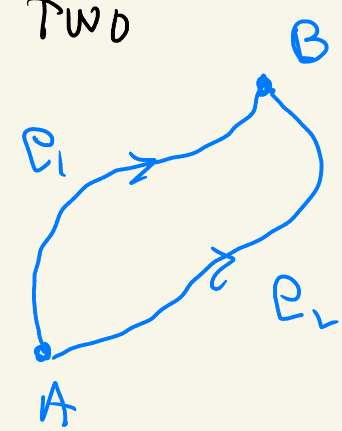
for every closed curve $\mathcal{C}$.

Turns out it goes the other way:

Theorem: If $\oint_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = 0$ for every closed curve $\mathcal{C}$, then $\vec{F}$ is conservative.

"Proof:" Assume $\oint_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = 0$ for every closed $\mathcal{C}$. Then

(1) $\int_{\mathcal{C}_1} \vec{F}\cdot\vec{T}\,ds = \int_{\mathcal{C}_2} \vec{F}\cdot\vec{T}\,ds$ for any two curves taking $A \to B$



I.e. $\mathcal{C}_2 - \mathcal{C}_1$ is closed so

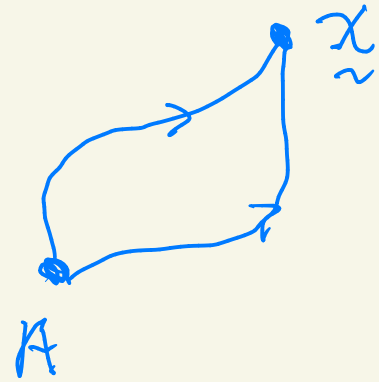
$$0 = \int_{\mathcal{C}_2-\mathcal{C}_1} \vec{F}\cdot\vec{T}\,ds = \int_{\mathcal{C}_1} \vec{F}\cdot\vec{T}\,ds - \int_{\mathcal{C}_2} \vec{F}\cdot\vec{T}\,ds$$

$$\Rightarrow \int_{\mathcal{C}_1} \vec{F}\cdot\vec{T}\,ds = \int_{\mathcal{C}_2} \vec{F}\cdot\vec{T}\,ds$$

(2) Define: $f(\underset{\sim}{x}) = \int_A^{\underset{\sim}{x}} \vec{F} \cdot \vec{T} \, ds$

This is well defined because the result is the same for every path $A \to \underset{\sim}{x}$

(So take any curve from $A \to \underset{\sim}{x} = (x, y, z)$)



(3) Turns out: $\nabla f = \vec{F}$.

(We skip this part of proof)

Problem: This is important for theoretical reason's (Complex Variables - how to put $i = \sqrt{-1}$ into Calculus - Math 185A)

... But doesnt help us answer the question

Q1: Given $\vec{F}$, how do we know its conservative?

We now address Q1

Q1: Given $\vec{F} = (M(\underset{\sim}{x}), N(\underset{\sim}{x}), P(\underset{\sim}{x}))$ how do we dermine whether or not $\vec{F}$ is conservative?

There is an extraordinay answer!

To get the answer, consider following diagram:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\nabla"] & \mathbb{R}^3 \arrow[r, "\text{Curl}"] & \mathbb{R}^3 \arrow[r, "\text{Div}"] & \mathbb{R} \\
\mathbb{R}^3 \arrow[u, "f"] & \mathbb{R}^3 \arrow[u, "\vec{F}"] & \mathbb{R}^3 \arrow[u, "\vec{F}"] & \mathbb{R}^3 \arrow[u, "f"]
\end{tikzcd}$$

The diagram Describes the functions which are the inputs and outputs of operators ∇, Curl, Div

Eg:

$$\begin{tikzcd}
\mathbb{R} & & \mathbb{R}^3 \\
\mathbb{R}^3 \arrow[u, "f"'] \arrow[r, "\nabla"] & \nabla f = \vec{F} & \mathbb{R}^3 \arrow[u]
\end{tikzcd}$$

The gradient inputs $f: \mathbb{R}^3 \to \mathbb{R}$

The gradient ouputs vector field $\nabla f = \vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$

Fact: (Not hard to check, but profoundly interesting)

Thm: If you do two in a row you get zero!

i.e. $\underset{\text{all } f}{\text{Curl}(\nabla f) = 0}$ and $\underset{\text{all } \vec{F}}{\text{Div}(\text{Curl}\,\vec{F}) = 0}$

Example: Show $\operatorname{Curl}(\nabla f) = 0$ for all f

Soln: $\operatorname{Curl}(\nabla f) = \operatorname{Curl}(f_x, f_y, f_z)$

$$= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix}$$

$$= \underset{\sim}{i}\,(\underbrace{\partial_y f_z - \partial_z f_y}_{f_{zy} - f_{yz} = 0}) - \underset{\sim}{j}\,(\underbrace{\partial_x f_z - \partial_z f_x}_{f_{yx} - f_{xz} = 0}) + \underset{\sim}{k}\,(\underbrace{\partial_x f_y - \partial_y f_x}_{f_{yx} - f_{xy} = 0})$$

$= 0$ ✓

Example: Show $\operatorname{Div}(\operatorname{Curl} \vec{F}) = 0 \quad \forall\, \vec{F}$

(Exam Problem?)

So back to our question:

Q1: Given $\vec{F}$, how do we know its conservative, i.e., how do we know $\exists f$ st $\nabla f = \vec{F}$?

We know: If $\nabla f = \vec{F}$, then $\text{Curl}\,\vec{F} = \text{Curl}\,\nabla f = 0$.

Does it go the other way?

I.e. is it true that if $\text{Curl}\,\vec{F} = 0$, then $\vec{F} = \nabla f$? some f.

If so we have a quick way of determining whether $\vec{F}$ is conservative – just take its Curl!

Ans: Yes, but only so long as $\vec{F}$ has No Singularities.

The precise therem is –

Theorem: If $\text{Curl}\,\vec{F} = 0$ and $\vec{F}$ is singularity free in a simply connect Domain $D \subseteq \mathbb{R}^3$, then $\oint_C \vec{F} \cdot \vec{T}\, ds = 0$ for every closed C & $\vec{F} = \nabla f$, $f(\underset{\sim}{x}) = \int_A^{\underset{\sim}{x}} \vec{F} \cdot \vec{T}\, ds$

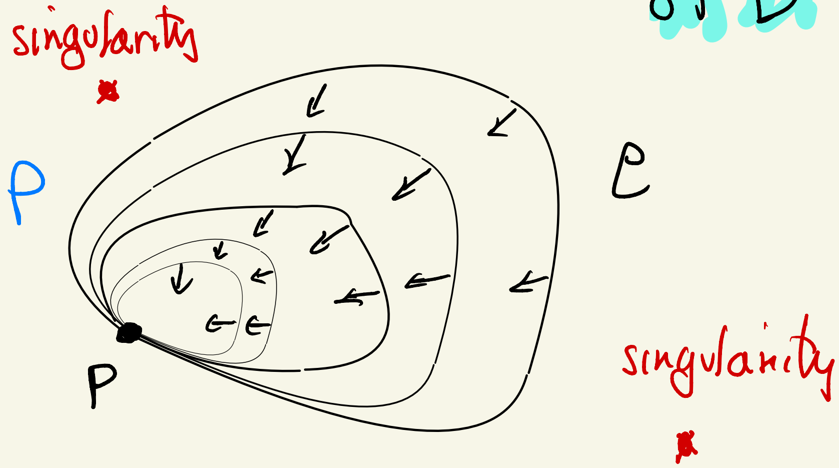
We need to define simply connected domain D

A simply connected domain is one that has

NO HOLES

Defn: $D \subseteq \mathbb{R}^3$ is simply connected if every closed curve in D can be continuously contracted to a point on C without passing out of D

Picture: Continuously contracting C to point P



Famous Example:

$$\underset{\sim}{F} = \overrightarrow{\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)} \qquad \underset{\sim}{x} = (x, y)$$

$\vec{F}$ is defined for $\underset{\sim}{x} = (x,y) \neq (0,0)$.

If $\text{Curl}\,\underset{\sim}{F} = 0$ $\underset{\sim}{x} \neq 0$ we cannot apply the theorem because $D = \{ \underset{\sim}{x} \in \mathbb{R}^2 : \underset{\sim}{x} \neq 0 \}$ is Not Simply Connected...

Because $\vec{F} = \overrightarrow{\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)}$ not defined $x = y = 0$,

...so D not simply connected!

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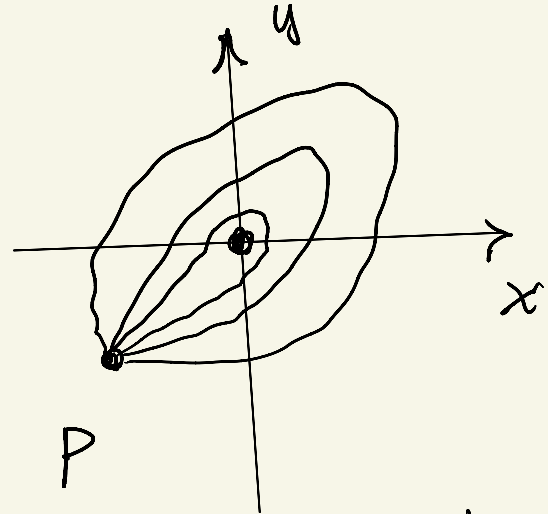
Picture: $\vec{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ $\underset{\sim}{x} \neq 0$

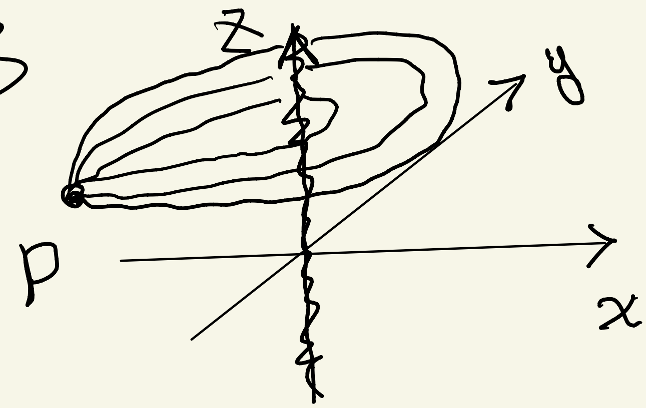
$D = \{ \underset{\sim}{x} \in \mathbb{R}^2 : \underset{\sim}{x} \neq 0 \}$



Not simply connected because there are closed curves in D which cannot be contracted to a point without passing thru $\underset{\sim}{x} = (0,0)$ and hence "out of D"

Similarly: $\vec{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$

$D = \{ \underset{\sim}{x} \in \mathbb{R}^3 : x \neq 0, y \neq 0 \}$



Check: $\text{Curl}\,\vec{F} = 0$ $\quad \vec{F} = \left(-\frac{y}{r^2}, \frac{x}{r^2}\right)$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & 0 \end{vmatrix} = \hat{i}\cdot 0 - \hat{j}\cdot 0 + (N_x - M_y)\hat{k}$$

$$N_x = \frac{\partial}{\partial x}\frac{x}{r^2} = \frac{1}{r^2} - 2\frac{x}{r^3}\frac{x}{r}$$

$$M_y = -\frac{\partial}{\partial y}\frac{y}{r^2} = -\frac{1}{r^2} + 2\frac{y}{r^3}\frac{y}{r}$$

$$N_x - M_y = \underbrace{\left(\frac{1}{r^2} + \frac{1}{r^2}\right)}_{\frac{2}{r^2}} - \underbrace{2\frac{1}{r^4}(x^2+y^2)}_{=\frac{2}{r^2}}$$

$$= 0$$

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Thus: $\text{Curl}\,\vec{F} = 0$ in D. We show $\int_C \vec{F}\cdot\vec{T}\,ds \neq 0$ on all closed curves. I.e., choose

C: unit circle $\vec{r}(t) = \overrightarrow{(\cos t, \sin t)}$

$0 \le t \le 2\pi$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_0^{2\pi} \vec{F}\cdot\vec{v}\,dt$$

r=1 on unit circle

$$= \int_0^{2\pi} \overrightarrow{\left(\frac{-y}{r^2}, \frac{x}{r^2}\right)} \cdot \overrightarrow{(-\sin t, \cos t)}\,dt$$

$$= \int_0^{2\pi} \overrightarrow{(-\sin t, \cos t)} \cdot \overrightarrow{(-\sin t, \cos t)}\,dt$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t\,dt = 2\pi$$

$\neq 0$!

Conclude: $\overrightarrow{F} = \overrightarrow{\left(\frac{-y}{r^2}, \frac{x}{r^2}, 0\right)}$

is curl free ($\text{Curl}\,\vec{F} = 0$) on its domain $x \neq 0, y \neq 0$, but it can't be conservative because $\oint_C \vec{F} \cdot \vec{T}\, ds \neq 0$ for some closed C.

Problem: To conclude $\text{Curl}\,\vec{F} = 0$ implies $\vec{F} = \nabla f$, we must have $\text{Curl}\,\vec{F} = 0$ on a simply connected Domain – the domain being all the values of (x, y, z) where $\text{Curl}\,\vec{F}$ can be computed, and $\text{Curl}\,\vec{F}(x, y, z) = 0$.

Example: Consider Newton's Force Field $\vec{F} = -\frac{\vec{r}}{r^3}$

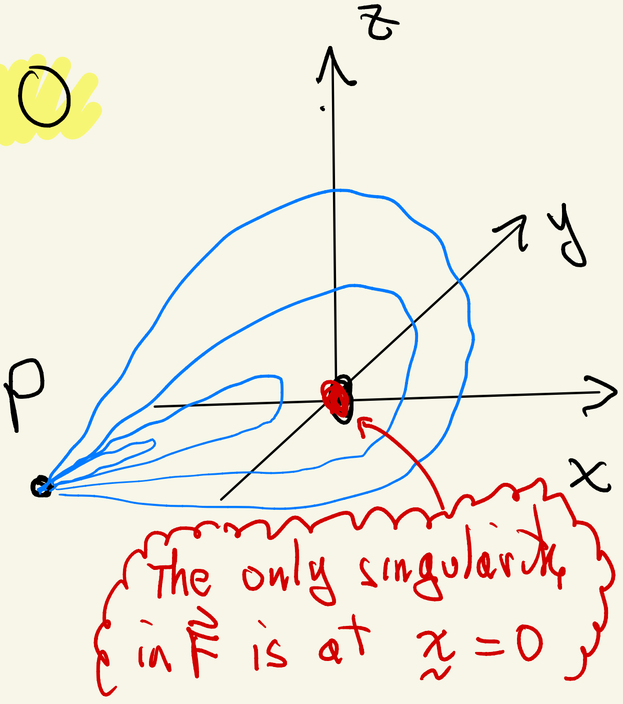
We know its conservative:

$$\nabla \frac{1}{r} = -\frac{\vec{r}}{r^3} = \vec{F}$$

Q: Is $\vec{F}$ curl free on a simply connected region? Ans Yes

$\vec{F}$ is defined everywhere except $\vec{r} = (x, y, z) = 0$

"You can always pull a closed curve around origin in $\mathbb{R}^3$"



Since $\vec{F} = \nabla \frac{1}{r}$ we know

$$\text{Curl}\,\vec{F} = \text{Curl}(\nabla f) = 0$$

because "Curl after ∇ is zero"

$$\mathbb{R} \xrightarrow{\nabla} \mathbb{R}^3 \xrightarrow{\text{Curl}} \mathbb{R}^3 \xrightarrow{\text{Div}} \mathbb{R}$$

We now check directly —

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ -\frac{x}{r^3} & -\frac{y}{r^3} & -\frac{z}{r^3} \end{vmatrix}$$

$$= \underset{\sim}{i}\underbrace{\left[\left(-\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right)\right) + \frac{\partial}{\partial z}\left(\frac{y}{r^3}\right)\right]}_{=0} - \underset{\sim}{j}\left[-\frac{\partial}{\partial x}\left(\frac{z}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{x}{r^3}\right)\right]$$

$$+ \underset{\sim}{k}\left[\frac{\partial}{\partial x}\left(-\frac{y}{r^3}\right) - \frac{\partial}{\partial y}\left(-\frac{x}{r^3}\right)\right]$$

Check: $\underset{\sim}{i}\underbrace{\left[\left(-\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right)+\frac{\partial}{\partial z}\left(\frac{y}{r^3}\right)\right)\right]}_{=0}$

$$\frac{\partial}{\partial y}\frac{z}{r^3} = -3\frac{z}{r^4}\frac{y}{r} \qquad \frac{\partial r}{\partial x} = \frac{x}{r} \text{ etc}$$

$$\frac{\partial}{\partial z}\frac{y}{r^3} = -3\frac{y}{r^4}\frac{z}{r}$$

$$-\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right)+\frac{\partial}{\partial z}\left(\frac{y}{r^3}\right) = 3\frac{zy}{r^5} - 3\frac{yz}{r^5} = 0 \checkmark$$

Similarly for $\underset{\sim}{j}$ & $\underset{\sim}{k}$ components.

Conclude: Newton Force $\vec{F} = -\frac{\vec{r}}{r^3}$ Satisfies $\text{Curl}\vec{F} = 0$ on a Simply connected Domain $\Rightarrow$ Conservative

Thm

Q: why is it true that if $\operatorname{curl}\vec{F} = 0$ on a simply connected domain then $\vec{F}$ is conservative?

<u>Ans</u>: Stokes Theorem! I.e. assume so.

Then:

(1) given $\mathcal{C}$ closed, it can be contracted to a point to form a surface $\mathcal{S}$ with $\mathcal{C}$ as boundary

(2) Stokes Thm: $\displaystyle\iint_{\mathcal{S}} \underbrace{\cancel{\operatorname{curl}\vec{F}}}_{0}\cdot\vec{n}\,dS = \int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds$

(3) Since $\displaystyle\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = 0$ for every closed curve $\mathcal{C}$, we know line integral is independent of path so $f(\underline{x}) = \displaystyle\int_A^{\underline{x}} \vec{F}\cdot\vec{T}\,ds$ is well defined

(4) $\nabla f = \vec{F}$ by previous Thm, so $\vec{F}$ conservative!